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SOME GENERAL PROPERTIES OF THE EQUATIONS OF VISCOELASTIC INCOMPRESSIBLE FLUID DYNAMICS

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We consider a system of equations describing the flows of an incompressible viscoelastic medium with a rheological equation of state containing derivatives of the stress tensor with respect to time. The initial system of equations for two-constant models of the medium is a quasilinear first-order system. Correct formulation of the problem under initial conditions requires the imposition of certain restrictions on the system matrix [1]. These restrictions, which are necessary to ensure the evolutionary character of the system, are imposed on the stress tensor in our case. We shall concentrate on one-dimensional motions for which the requirement of evolutionary character renders the system hyperbolic. It is then possible to indicate sufficient conditions which ensure the uniqueness of the continuous solution of the one-dimensional steadystate boundary value problem.

Hyperbolic systems of equations of viscoelastic fluid dynamics have discontinuous solutions for certain models (e.g. that of Oldroyd [2]). Discontinuous flows of materials with memory in which the stresses are functionals of their "strain history" are discussed in [3]. We shall consider the discontinuities in Oldroyd's model when the differential relationship between the stress tensors and straining rates is given. A necessary condition for the existence of discontinuities is formulated. The problem of evolution of a velocity jump in one-dimensional motion is considered.

1. The conditions of evolutionary character. Let a viscoelastic incompressible fluid move in a plane channel $0 < z < \delta$ or in a half-space $z > 0$. We assume that all the parameters of motion except the pressure are functions of the single space coordinate z and of the time t.

The equations of motion are in this case of the form

$$\begin{gathered}
\frac{\partial v_x}{\partial t} + v_z \frac{\partial v_x}{\partial z} - \frac{1}{\rho}\frac{\partial T_{xz}}{\partial z} - \frac{1}{\rho} P_x - F_x = 0 \\
\frac{\partial v_y}{\partial t} + v_z \frac{\partial v_y}{\partial z} - \frac{1}{\rho}\frac{\partial T_{yz}}{\partial z} - \frac{1}{\rho} P_y - F_y = 0 \\
\frac{\partial v_z}{\partial t} - \frac{1}{\rho}\frac{\partial T_{zz}}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} - F_z = 0
\end{gathered} \qquad (1.1)$$

Here $P_x(t) = -\partial p / \partial x$ and $P_y(t) = -\partial p/\partial y$ can be regarded as given functions. The last equation of (1.1) makes use of the incompressibility condition

$$\partial v_z/\partial z = 0, \qquad v_z = v_z(t)$$

The components of the stress tensor are interrelated by six rheological equations. We shall consider:

Oldroyd's "contravariant" model,

$$T_{ij} + \lambda(\partial T_{ij}/\partial t + v_k T_{ij,k} - v_{i,k} T_{kj} - v_{j,k} T_{ik}) = \eta(v_{i,j} + v_{j,i}) \tag{1.2}$$

Oldroyd's "covariant" model,

$$T_{ij} + \lambda(\partial T_{ij}/\partial t + v_k T_{ij,k} + v_{k,j} T_{ik} + v_{k,i} T_{kj}) = \eta(v_{i,j} + v_{j,i}) \tag{1.3}$$

a model with a derivative in the sense of Jaumann,

$$T_{ij} + \lambda(\partial T_{ij}/\partial t + v_k T_{ij,k} - \omega_{ik} T_{kj} - \omega_{jk} T_{ik}) = \eta(v_{i,j} + v_{j,i}) \tag{1.4}$$

$$\omega_{ij} = {}^1/_2(v_{i,j} - v_{j,i})$$

Equations (1.2) - (1.4) are written out in Cartesian coordinates. The tensor T_{ij} is related to the total stress tensor p_{ij} by the equation $p_{ij} = -p\delta_{ij} + T_{ij}$.

The system consisting of the first two equations of (1.1) together with one of the groups of equations (1.2), (1.3), or (1.4) is closed with respect to the eight unknown functions v_x, v_y, T_{xx}, T_{xy}, T_{xz}, T_{yy}, T_{yz}, T_{zz} provided that $v_z\ (t)$ is a given function of time. The last equation of (1.1) then defines the transverse pressure distribution. The condition $v_z \neq 0$ means that fluid can be either injected or sucked out through the planes bounding the stream.

Let us introduce the vector function $\mathbf{f} = (v_x, v_y, T_{xx}, T_{xy}, T_{xz}, T_{yy}, T_{yz}, T_{zz})$. This enables us to write the complete system of equations for each model in matrix form (*),

$$\frac{\partial \mathbf{f}}{\partial t} + A(\mathbf{f})\frac{\partial \mathbf{f}}{\partial z} + \mathbf{b}(\mathbf{f}, z, t) = 0 \tag{1.5}$$

The matrix A for each of the systems in question is a nonsymmetric eighth-order matrix which depends linearly on the vector $\mathbf{f}$; the vector $\mathbf{b}$ is of the form

$$\mathbf{b} = \left(-\frac{1}{\rho}P_x - F_x, -\frac{1}{\rho}P_y - F_y, \frac{1}{\lambda}T_{xx}, \frac{1}{\lambda}T_{xy}, \frac{1}{\lambda}T_{xz}, \frac{1}{\lambda}T_{yy}, \frac{1}{\lambda}T_{yz}, \frac{1}{\lambda}T_{zz}\right)$$

Quasilinear system (1.5) must be evolutionary, i.e. the problem with initial conditions for this system must be correctly formulated. The example given in [4] indicates that the equations of a viscoelastic medium may be nonevolutionary; we therefore begin our analysis of system (1.5) by deriving the conditions of evolutionary character. Recalling the considerations presented in [1], we assume that the condition of realness of the roots of the characteristic equation

$$|A - \tau E| = 0 \tag{1.6}$$

where E is a unit matrix, is necessary for evolutionary character.

The roots of Eq. (1.6) for system (1.1), (1.2) have the following values

$$\tau_{1,2,3,4} = v_z, \quad \tau_{5,6} = v_z + \left[\frac{1}{\rho}\left(T_{zz} + \frac{\eta}{\lambda}\right)\right]^{1/2}, \quad \tau_{7,8} = v_z - \left[\frac{1}{\rho}\left(T_{zz} + \frac{\eta}{\lambda}\right)\right]^{1/2} \tag{1.7}$$

*) The matrix A can be multiplied by the vector $\mathbf{f}$ according to the rules of multiplication by a vector column.

Hence, the necessary condition of evolutionary character of system (1.1), (1.2) is of the form

$$T_{zz} + \eta/\lambda > 0 \tag{1.8}$$

Similarly, in considering Eq. (1.6) for system (1.1), (1.3) we obtain the following values for the roots:

$$\tau_{1,2,3,4} = v_z, \qquad \tau_{5,6,7,8} = v_z \pm \left[\frac{2\eta/\lambda - (T_{xx} + T_{yy}) \pm \sqrt{\Delta}}{2\rho}\right]^{1/2}$$
$$\Delta = (T_{yy} - T_{xx})^2 + 4T_{xy}^2 \tag{1.9}$$

System (1.1), (1.3) is therefore evolutionary only if

$$\eta/\lambda - {}^1/_2(T_{xx} + T_{yy} + \sqrt{\Delta}) > 0 \tag{1.10}$$

In the case $v_y = 0$, $T_{xy} = T_{yy} = T_{yz} = 0$ the condition (1.10) becomes

$$\eta/\lambda - T_{xx} > 0$$

This inequality differs from (1.8) in the fact that it contains $-T_{xx}$ instead of T_{zz}. This is a consequence of the fact that every solution of the problem of motion of the covariant model in the case $v_y = 0$, $T_{xy} = T_{yy} = T_{yz} = 0$ can be obtained from the solution for the contravariant model under the same assumptions concerning the motion by interchanging the quantities T_{zz} and $-T_{xx}$ in the latter solution.

On considering the characteristic equation for system (1.1), (1.4) we obtain the following expressions for the roots:

$$\tau_{1,2,3,4} = v_z, \quad \tau_{5,6,7,8} = v_z \pm \left[\frac{1}{2\rho}\left(\frac{2\eta}{\lambda} + T_{zz} - \frac{T_{xx} + T_{yy}}{2} \pm \sqrt{\Delta'}\right)\right]^{1/2}$$
$$\Delta' = {}^1/_4\Delta \tag{1.11}$$

The condition of evolutionary character of system (1.1), (1.4) which describes the one-dimensional motion of the model with a Jaumann derivative is of the form

$$T_{zz} - {}^1/_2(T_{xx} + T_{yy}) - \sqrt{\Delta'} + 2\eta/\lambda > 0 \tag{1.12}$$

Inequalities (1.8), (1.10), (1.12) for the corresponding media must be fulfilled throughout the flow region at all instants. If this is not the case, we must alter the rheological model at least in the region where the initial equations become nonevolutionary. The quantities which dictate the evolutionary character of a system must themselves be determined by solving a certain mixed problem. A system can therefore enter the nonevolutionary region because of unsuitably chosen initial or boundary conditions.

In the case of linear viscoelasticity the conditions of evolutionary character are fulfilled automatically. This means that nonevolutionary character is a consequence of the kinematic nonlinearity of the tensor derivative with respect to time.

2. Uniqueness of the solution of the mixed problem. Let us assume that the necessary conditions of evolutionary character are fulfilled. An important factor in proving uniqueness is the possibility of reducing the matrix A to diagonal form. Since the matrix A is nonsymmetric, it follows that it can be reduced to diagonal form if the maximum number of linearly independent eigenvectors of the matrix coincides with its order. We infer from (1.7), (1.9), (1.11) that characteristic equation (1.6) has multiple roots for each of the models under consideration. This requires direct consideration of the problem of the maximum number of independent linear eigenvectors which yields the required result for each of models (1.2)-(1.4): the required number is eight. Hence, the matrix A is reducible to diagonal form and system (1.5) is hyperbolic.

Let us consider the two solutions $\mathbf{f}_1$ and $\mathbf{f}_2$ which satisfy (1.5). We assume that $\mathbf{f}_1$ and $\mathbf{f}_2$ and their first derivatives are continuous in the domain R $(0 < t < T,\ 0 < z < \delta)$. Substituting first $\mathbf{f}_1$ and then $\mathbf{f}_2$ into (1.5) and subtracting the result of one substitution from the other, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{f}_1)\frac{\partial \mathbf{u}}{\partial z} + [A(\mathbf{f}_1) - A(\mathbf{f}_2)]\frac{\partial \mathbf{f}_2}{\partial z} + \mathbf{b}(z, t, \mathbf{f}_1) - \mathbf{b}(z, t, \mathbf{f}_2) = 0 \quad (2.1)$$

$$(\mathbf{u} = \mathbf{f}_1 - \mathbf{f}_2)$$

If the body force vector $\mathbf{F}$ either depends linearly on the velocity or is a given function of z and t, then the vector $\mathbf{b}$ depends linearly on $\mathbf{f}$. The matrix A also depends linearly on $\mathbf{f}$, so that we have the relations

$$A_{ij} = c_{ijk} f_k + d_{ij}, \qquad b_i = l_{ik} f_k + m_i$$

Here c_{ijk}, d_{ij} are constants and l_{ik}, m_i are functions of z and t.
Hence, we can rewrite (2.1) as

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{f}_1)\frac{\partial \mathbf{u}}{\partial z} + B\left(z, t, \frac{\partial \mathbf{f}_2}{\partial z}\right)\mathbf{u} = 0, \quad B_{ij} = c_{ikj}\frac{\partial f_{2k}}{\partial z} + l_{ij} \qquad (2.2)$$

Equation (2.2) can be regarded as a homogeneous linear equation in $\mathbf{u}$. Our subsequent constructions are based on the method described in [5] in connection with linear hyperbolic systems. Linear replacement of the unknown function reduces the matrix to diagonal form.

Let $\mathbf{u} = H\mathbf{U}$, where the columns of the matrix H consist of the linearly independent eigenvectors of the matrix A. Carrying out some simple transformations, we now obtain the following equation for $\mathbf{U}$:

$$\frac{\partial \mathbf{U}}{\partial t} + G\frac{\partial \mathbf{U}}{\partial z} + K\mathbf{U} = 0 \qquad (2.3)$$

Here $G = H^{-1}AH$ is a diagonal matrix whose diagonal consists of the eigenvalues of the matrix A, and

$$K = H^{-1}\left(B + \frac{\partial H}{\partial t} + A\frac{\partial H}{\partial z}\right)$$

Let us make yet another substitution, setting $\mathbf{U} = e^{\alpha t}\mathbf{W}$, where $\alpha > 0$. This implies that $\mathbf{W}$ satisfies the equation

$$\frac{\partial \mathbf{W}}{\partial t} + G\frac{\partial \mathbf{W}}{\partial z} + (K + \alpha E)\mathbf{W} = 0 \qquad (2.4)$$

where E is a unit matrix.

Let us take the scalar product of (2.4) and $2\mathbf{W}$ and make use of an identity valid for any vector $\mathbf{W}$ and any symmetric matrix G ,

$$2\left(G\frac{\partial \mathbf{W}}{\partial z}, \mathbf{W}\right) = \frac{\partial}{\partial z}(G\mathbf{W}, \mathbf{W}) - \left(\frac{\partial G}{\partial z}\mathbf{W}, \mathbf{W}\right)$$

This yields

$$\frac{\partial}{\partial t}(W^2) + \frac{\partial}{\partial z}(G\mathbf{W}, \mathbf{W}) + (M\mathbf{W}, \mathbf{W}) = 0, \qquad M = 2K - \frac{\partial G}{\partial z} + 2\alpha E \quad (2.5)$$

It is clear that if we choose a sufficiently large $\alpha > 0$, then the quadratic form $(M\mathbf{W},\mathbf{W})$ is positive-definite. Next, integrating (2.5) over the domain R, we obtain

$$\int_0^\delta W^2 dz \Big|_0^T + \int_0^T (G\mathbf{W}, \mathbf{W})_{z=\delta} dt - \int_0^T (G\mathbf{W}, \mathbf{W})_{z=0} dt + \int_0^T \int_0^\delta (M\mathbf{W}, \mathbf{W})\, dz dt = 0 \quad (2.6)$$

Let $\mathbf{W} = 0$ for $t = 0$, i.e. let an initial condition be imposed on $\mathbf{f}$, and let the inequalities

$$(G\mathbf{W}, \mathbf{W})_{z=0} \leqslant 0, \qquad (G\mathbf{W}, \mathbf{W})_{z=\delta} \geqslant 0 \qquad (2.7)$$

hold at the channel boundaries.

Equation (2.6) then implies that $\mathbf{W} \equiv 0$, and uniqueness has been proved.

For $z = 0$ we specify k boundary conditions linear in the vector $\mathbf{f}$,

$$(\mathbf{f}, \mathbf{q}_1) = \varkappa_1(t), \ldots, (\mathbf{f}, \mathbf{q}_k) = \varkappa_k(t)$$

Here $\mathbf{q}_1, \ldots, \mathbf{q}_k$ are linearly independent and can be functions of t; $\varkappa_i(t)$ are given functions.

These boundary conditions are homogeneous for the vector $\mathbf{u}$,

$$(\mathbf{u}, \mathbf{q}_1) = 0, \ldots, (\mathbf{u}, \mathbf{q}_k) = 0$$

Replacing $\mathbf{u}$ by its expression in terms of $\mathbf{W}$, we obtain the following boundary conditions for $\mathbf{W}$:

$$(\mathbf{W}, \mathbf{Q}_1) = 0, \ldots, (\mathbf{W}, \mathbf{Q}_k) = 0, \qquad \mathbf{Q}_i = H'\mathbf{q}_i \qquad (2.8)$$

Here H' is the transpose of the matrix H. The minimum number of conditions of the form (2.8) for which $(G\mathbf{W}, \mathbf{W})_{z=0} \leqslant 0$, is equal to the number of positive eigenvalues of the matrix $A(\mathbf{f}_1)$, including multiple eigenvalues [5]. Similarly, the minimum number of boundary conditions of the form (2.8) for $z = \delta$ is equal to the number of negative eigenvalues of the matrix $A(\mathbf{f}_1)$, including multiple eigenvalues.

We note that if the number k is known, then the choice of the independent vectors $\mathbf{q}_i$ is not completely arbitrary. In order for the quadratic form $(G\mathbf{W}, \mathbf{W})$ to have a fixed sign, the components of the vectors $\mathbf{Q}_i$, and therefore the components of $\mathbf{q}_i$, must conform to certain conditions. In fact, considering (2.8) as a system of linear equations in the components of the vector $\mathbf{W}$, we obtain

$$\mathbf{W} = \gamma_1 \mathbf{\Psi}_1(\mathbf{Q}_1, \ldots, \mathbf{Q}_k) + \ldots + \gamma_{n-k} \mathbf{\Psi}_{n-k}(\mathbf{Q}_1, \ldots, \mathbf{Q}_k)$$

Here the vectors $\mathbf{\Psi}_1, \ldots \mathbf{\Psi}_{n-k}$ form the fundamental system of solutions; in our case $n = 8$. The quadratic form $(G\mathbf{W}, \mathbf{W})$ is now reducible to a form defined on the $(n-k)$-dimensional vectors $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{n-k})$,

$$(G\mathbf{W}, \mathbf{W}) = (G\mathbf{\Psi}_i, \mathbf{\Psi}_j)\gamma_i\gamma_j$$

In order to ensure that the latter quadratic form is of fixed sign on all the vectors $\boldsymbol{\gamma}$, we must impose certain conditions on the corner minors of the matrix $\| (G\mathbf{\Psi}_i, \mathbf{\Psi}_j) \|$ which depends on the vectors $\mathbf{Q}_i$.

The characteristics of quasilinear systems of equations themselves depend on the boundary conditions, and their slope is not known in advance. However, in certain cases, e.g. for systems (1.1), (1.2) or (1.1), (1.3) which describe the flow for Oldroyd's models it is possible to determine the slope of the characteristics by solving the equation for T^{*}_{zz} in the case of model (1.2) or the equations for T_{xx}, T_{xy} and T_{yy} in the case of model (1.3). All of these functions satisfy the same equation

$$L(T)\equiv\left(\frac{\partial}{\partial t}+v_z(t)\frac{\partial}{\partial z}+\frac{1}{\lambda}\right)T=0 \tag{2.9}$$

whose solution is of the form

$$T(z,\,t)=e^{-t/\lambda}F\left(z-\int\limits_0^t v_z(t')\,dt'\right)$$

In this case Formulas (1.7) or (1.9) can be used to compute all the characteristic roots $\tau_i(z,\,t)$. System (1.1), (1.4) has stronger nonlinearity than do systems (1.1), (1.2) or (1.1), (1.3). It is therefore difficult in this case to find the characteristic roots as functions of z and t in explicit form.

For example, let us consider the uniqueness of the solution of the following problem. Let a viscoelastic fluid corresponding to model (1.2) flow in a channel with impermeable walls. Let us assume that $v_y=0$, $T_{xy}=T_{yy}=T_{yz}=0$. The quantity T_{zz} satisfies (2.9) and is in this case given by $T_{zz}=T_0(z)\,e^{-t/\lambda}$. We now have the following system of linear equations for the quantities v_x and T_{xz} :

$$\frac{\partial v_x}{\partial t}-\frac{1}{\rho}\frac{\partial T_{xz}}{\partial z}-\frac{1}{\rho}P_x-F_x=0,\qquad \frac{\partial T_{xz}}{\partial t}-c^2\frac{\partial v_x}{\partial z}+\frac{1}{\lambda}T_{xz}=0$$

The unknown vector has two components, $\mathbf{f}=(v_x,\,T_{xz})$. The characteristic values are

$$\tau_{1,2}=\pm c,\qquad c=\left[\frac{1}{\rho}\left(T_{zz}+\frac{\eta}{\lambda}\right)\right]^{1/2}$$

For the quadratic form $(G\mathrm{W},\mathrm{W})$ we obtain

$$(G\mathrm{W},\mathrm{W})=e^{-2\alpha t}\,(v_{1x}-v_{2x})(T_{2xz}-T_{1xz})\,/\,\rho$$

In view of the above considerations we must specify one boundary condition for $z=0$ and one for $z=\delta$. It is clear that if we specify the condition of adhesion at each wall, namely $v_{1x}=v_{2x}=v_x$, then $(G\mathrm{W},\,\mathrm{W})$ vanishes and the solution is unique. The solution of this problem in the case $T_0(z)=\text{const}$, $F_x=0$ under the boundary conditions $v_x(0)=v_x(\delta)=0$ is given in [4].

The latter paper does not deal with the uniqueness of the generalized solution of system (1.5). This problem is of interest, since strong discontinuities are possible in a viscoelastic medium.

3. Discontinuous flows of an Oldroyd fluid. In order to find the relations at the discontinuities we reduce the "principal part" of the system of equations of three-dimensional flow for Oldroyd's contravariant model to "divergent" form. The system of equations of motion together with the system of rheological equations transformed with the aid of the equations of motion and the continuity equation becomes

$$\frac{\partial}{\partial t}(\rho v_i)+(\rho v_i v_k+p\delta_{ik}-T_{ik})_{,\,k}=\rho F_i$$

$$\frac{\partial}{\partial t}(T_{ij}+\rho v_i v_j)+\left[(T_{ij}v_k-T_{kj}v_i-T_{ik}v_j+\rho v_i v_j v_k)-\frac{\eta}{\lambda}\,(v_i\delta_{kj}+v_j\delta_{ki})\right]_{,\,k}=$$

$$= -\frac{1}{\lambda} T_{ij} + v_i(-p_{,j} + \rho F_j) + v_j(-p_{,i} + \rho F_i) \qquad (3.1)$$

Since the fluid is incompressible, only tangential discontinuities for which $\{v_n\} = 0$ are possible, so that $\{p\} = \{T_{nn}\}$.

Let us consider the discontinuities such that $\{T_{nn}\} = 0$. The pressure then varies continuously through the discontinuity surface. We also assume that the stresses T_{ij}, the pressure gradient, and the external body forces are bounded in the neighborhood of the discontinuity surface. Integrating Eqs. (3.1) over a narrow layer containing the discontinuity surface defined by the equation $\varphi(x_i, t) = 0$ and making the layer thickness go to zero, we obtain

$$\partial\varphi / \partial t \{\rho v_i\} + \varphi_{,k} \{\rho v_i v_k - T_{ik}\} = 0 \qquad (3.2)$$

$$\partial\varphi / \partial t \{T_{ij} + \rho v_i v_j\} + \varphi_{,k} \{T_{ij} v_k - T_{kj} v_i - T_{ik} v_j + \rho v_i v_j v_k - \frac{\eta}{\lambda}(v_i \delta_{kj} + v_j \delta_{kj})\} = 0$$

Let us divide Eqs. (3.2) by $|\nabla\varphi|$ and multiply the last equation of (3.2) by n_j, where $n_j = \varphi_{,j} / |\nabla\varphi|$ are the components of the unit vector of the normal to the discontinuity surface. Carrying out contraction over the subscript j, we obtain

$$-\rho c \{v_i\} = \{T_{ik}\} n_k, -(\rho v_n \{v_i\} - \{T_{ij}\} n_j)\, c = v_n \{T_{ij}\} n_j - (T_{nn} + \eta / \lambda) \{v_i\}$$

$$c = \left(v_n + \frac{1}{|\nabla\varphi|}\frac{\partial\varphi}{\partial t}\right)$$

where c is the velocity of propagation of the discontinuity surface relative to the medium. The latter equations imply that

$$\{T_{ij} n_j\}^2 = (T_{nn} + \eta / \lambda)\, \rho \{v_i\}^2$$

for $\{V\} \neq 0$. This is possible only if

$$T_{nn} + \eta/\lambda > 0 \qquad (3.3)$$

If the inequality $T_{nn} + \eta/\lambda < 0$ is fulfilled throughout some region occupied by the moving liquid, then the above statements imply that velocity discontinuity surfaces cannot exist in the flow.

We note that in the case of one-dimensional motion the condition (3.3) coincides with (1.8) which is necessary if the system is to be evolutionary. The initial system of equations is then hyperbolic.

We can show that fulfillment of (3.3) is necessary in order for the system of equations of three-dimensional flow of model (1.2) to be evolutionary. The direction of the vector $\mathbf{n}$ must be arbitrary in this case.

If (3.3) is fulfilled, then the velocity of propagation of the discontinuity surface is given by

$$c = \pm [(T_{nn} + \eta/\lambda) / \rho]^{1/2}$$

The function $\varphi(x_i, t)$ which vanishes at the discontinuity surface satisfies the "eikonal equation"

$$(\partial\varphi / \partial t)^2 = (V\nabla\varphi \pm \sqrt{T_{ij}\varphi_{,i}\varphi_{,j}/\rho + \eta(\nabla\varphi)^2/\rho\lambda})^2 \qquad (3.4)$$

We can show by conventional methods that the strong discontinuity surface coincides with the characteristic surfaces of the initial system of equations. This statement, which is always valid for linear systems is in this case a consequence of the "weak" nonlinearity of the system of equations under consideration.

Let us assume that $\{\mathbf{V}\} = 0$, so that $\{T_{ij}\}\ n_j = 0$ and therefore $\{\mathbf{p}_n\} = 0$. In this case $c = 0$ and the quantity $\mathbf{p}_{n'}$ can experience a discontinuity at such a surface which moves together with the particles constituting the medium. This quantity represents the stress acting on an area with the normal $\mathbf{n}'$, where $n_j n_j' = 0$. This fact can be verified by multiplying the last equation (3.2) by n_j' and contracting over j.

Thus, two types of discontinuities are possible in the medium under consideration under condition (3.3). A discontinuity surface of the first type propagates with a nonzero velocity relative to the medium and represents the locus of jumps in the tangential component of the velocity and in the components of the stress tensor. A surface of the second type does not propagate relative to the medium, and constitutes the locus of jumps in the components of the stresses at the areas orthogonal to the discontinuity surface with the exception of the component of this vector in the direction $\mathbf{n}$. The component along this direction is continuous, since the symmetry of the stress tensor implies that $\{\mathbf{p}_{n'}\}\,\mathbf{n} = \{\mathbf{p}_n\}\,\mathbf{n}' = 0$.

In contrast to gas dynamic flows, discontinuities in nonsteadystate motions of a viscoelastic fluid cannot arise out of continuous flows; they represent the development of some initial discontinuity. This fact is established below for the case of one-dimensional motion.

Now let us consider the special case of one-dimensional flow in a plane channel or in a half-space. The discontinuities occur at the characteristics of system (1.1), (1.2). The characteristics of the first type at which velocity jumps can occur satisfy the differential equations

$$\frac{dz}{dt} = v_z(t) \pm c, \qquad c = \left[\frac{1}{\rho}\left(T_{zz} + \frac{\eta}{\lambda}\right)\right]^{1/2} \tag{3.5}$$

The characteristics of the second type, at which only the quantities T_{xx}, T_{xy}, T_{yy}, can experience discontinuities, satisfy the equation

$$dz\,/\,dt = v_z(t) \tag{3.6}$$

Applying (3.2) to the case of one-dimensional flow, we can show that the following relations hold at the characteristics of the first type:

$$\{T_{xx}\} = \rho\{v_x\}^2 \mp 2T_{xz}^{(1)}\,\{v_x\}\,/\,c, \qquad \{T_{yy}\} = \rho\{v_y\}^2 \mp 2T_{yz}^{(1)}\{v_y\}/\,c$$

$$\{T_{xy}\} = \rho\,\{v_x\}\,\{v_y\} \mp (T_{xz}^{(1)}\,\{v_y\} + T_{yz}^{(1)}\,\{v_x\})\,/\,c$$

$$\{T_{xz}\} = \mp\,\rho c\,\{v_x\}, \qquad \{T_{yz}\} = \mp\,\rho c\{v_y\} \tag{3.7}$$

Relations (3.7) imply that all quantities are continuous at the first-type characteristics provided that $\{\mathbf{V}\} = 0$.

In order to investigate the evolution of the jump $\{v_x\}$ let us consider the following system of two equations closed with respect to the quantities v_x and T_{xz} :

$$\begin{aligned} &\frac{\partial v_x}{\partial t} + v_z\,\frac{\partial v_x}{\partial z} - \frac{1}{\rho}\,\frac{\partial T_{xz}}{\partial z} - \frac{1}{\rho}\,P_x - F_x = 0 \\ &\frac{\partial T_{xz}}{\partial t} - \left(T_{zz} + \frac{\eta}{\lambda}\right)\frac{\partial v_x}{\partial z} + v_z\,\frac{\partial T_{xz}}{\partial z} + \frac{1}{\lambda}\,T_{xz} = 0 \end{aligned} \tag{3.8}$$

Introducing the unknown vector function $\mathbf{f} = (v_x, T_{xz})$, we can rewrite (3.8) in the form

$$\frac{\partial \mathbf{f}}{\partial t} + A\frac{\partial \mathbf{f}}{\partial z} + B\mathbf{f} + \mathbf{g} = 0$$

By linear replacement of the unknown function $\mathbf{u} = H^{-1}\mathbf{f}$,

$$H^{-1} = \frac{1}{2}\begin{Vmatrix} 1 & -1/\rho c \\ 1 & 1/\rho c \end{Vmatrix}, \qquad \mathbf{u} = \frac{1}{2}\left(v_x - \frac{T_{xz}}{\rho c},\ v_x + \frac{T_{xz}}{\rho c}\right)$$

we can reduce system (3.8) to diagonal form,

$$\frac{\partial \mathbf{u}}{\partial t} + G\frac{\partial \mathbf{u}}{\partial z} + S\mathbf{u} + \mathbf{h} = 0 \tag{3.9}$$

Here $\mathbf{h}$ does not depend on $\mathbf{u}$, and the matrices G and S are of the form

$$G = \begin{Vmatrix} v_z + c & 0 \\ 0 & v_z - c \end{Vmatrix}$$

$$S = \frac{1}{2}\begin{Vmatrix} \frac{1}{\lambda} + \frac{\partial \ln c}{\partial t} + \left(1 + \frac{v_z}{c}\right)\frac{\partial c}{\partial z} & -\frac{1}{\lambda} - \frac{\partial \ln c}{\partial t} - \left(1 + \frac{v_z}{c}\right)\frac{\partial c}{\partial z} \\ -\frac{1}{\lambda} - \frac{\partial \ln c}{\partial t} + \left(1 - \frac{v_z}{c}\right)\frac{\partial c}{\partial z} & \frac{1}{\lambda} + \frac{\partial \ln c}{\partial t} - \left(1 - \frac{v_z}{c}\right)\frac{\partial c}{\partial z} \end{Vmatrix}$$

Applying the jump-taking operation to Eq. (3.9), we find that

$$\frac{D_+}{Dt}\{v_x\} = F_+(z, t)\{v_x\},\ F_+(z, t) = -\frac{1}{2}\left[\frac{1}{\lambda} + \frac{\partial \ln c}{\partial t} + \left(1 + \frac{v_z}{c}\right)\frac{\partial c}{\partial z}\right] \tag{3.10}$$

at the characteristic $dz/dt = v_z + c$.
Here

$$\frac{D_+}{Dt} = \frac{\partial}{\partial t} + (v_z + c)\frac{\partial}{\partial z}$$

is the operator of total differentiation with respect to time along the characteristic. At this characteristic we have $\{u_2\} = 0$. Integrating (3.10) under the initial condition $\{v_x\}|_{t=t_0} = \{v_x\}_0$, we obtain

$$\{v_x\} = \{v_x\}_0 \exp\int_{t_0}^{t} F_+(z_+(t'), t')\,dt' \tag{3.11}$$

In integrating F_+ we must replace z by $z_+(t')$ in (3.10), where $z_+(t')$ is the solution of the differential equation

$$\frac{dz_+}{dt'} = v_z(t') + c(z_+, t')$$

under the initial condition $z_+ = z_0$ for $t' = t_0$.
Similarly, at the characteristic $dz/dt = v_z - c$ we obtain the equation

$$\frac{D_-}{Dt}\{v_x\} = F_-(z,t)\{v_x\}, \quad F_-(z,t) = -\frac{1}{2}\left[\frac{1}{\lambda} + \frac{\partial \ln c}{\partial t} + \left(\frac{v_z}{c} - 1\right)\frac{\partial c}{\partial z}\right]$$

whose solution is of the form

$$\{v_x\} = \{v_x\}_0 \exp \int_{t_0}^{t} F_-(z_-(t'), t')\,dt', \quad \frac{dz_-}{dt'} = v_z(t') - c(z_-, t'), \quad z_-(t_0) = z_0 \quad (3.12)$$

Formulas which coincide exactly with (3.11), (3.12) yield an expression for $\{v_y\}$. Expressions (3.7) can then be used to find all the remaining quantities provided the stresses $T_{xx}^{(1)}, T_{xy}^{(1)}, T_{yy}^{(1)}$ to one side of the line of discontinuity are given.

The quantities $\{T_{xx}\}, \{T_{xy}\}, \{T_{yy}\}$ at characteristics of the second type which satisfy (3.6) are given by the formulas

$$\{T_{xx}\} = \{T_{xx}\}_0 \exp\frac{-t}{\lambda}, \quad \{T_{xy}\} = \{T_{xy}\}_0 \exp\frac{-t}{\lambda}, \quad \{T_{yy}\} = \{T_{yy}\}_0 \exp\frac{-t}{\lambda}$$

Let us consider the special case where c depends only on t. We then have

$$\{V\} = \{V\}_0 \left[\left(T_0 + \frac{\eta}{\lambda}\right)\Big/\left(T_0 \exp\frac{-t}{\lambda} + \frac{\eta}{\lambda}\right)\right]^{1/4} \exp\frac{-t}{2\lambda}, \quad T_{zz} = T_0 \exp\frac{-t}{\lambda}$$

at the characteristics of the first type.

This formula indicates that the velocity jump decreases monotonically with increasing t, and tends to zero as $t \to \infty$.

It can be shown that in the other special case where c depends only on z the velocity discontinuity also goes to zero as $t \to \infty$. However, the initial discontinuity can increase for small times if $T_* = T_{zz}|_{z=0}$ is large enough. For large T_* and small t we have the asymptotic formula

$$\{V\} \approx \{V\}_0 \exp\left(\frac{1}{2}\left[1 - \exp\left(\mp\sqrt{\frac{T_*}{\rho}}\frac{t}{2\lambda v_0}\right)\right]\right) \qquad v_0 = v_z \quad z_+(0) = 0$$

For example, if $v_0 > 0$, then the initial discontinuity at the characteristic $z_+(t)$ increases over a small time interval. Nevertheless, the magnitude of the discontinuity remains finite as T_* tends to infinity.

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